ON THE DETERMINATION OF THE SURFACE OF BULGING MATERIAL CAUSED BY THE PENETRATION OF A THIN BLADE INTO A PLASTIC HALF-SPACE

(OB OPREDELENII POVERKHNOSTI VYPUCHIVSHEGOSIA MATERIALA PRI VDAVLENII TONKOGO LEZVIIA V PLASTICHESKOE POLUPROSTRANSTVO)

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The plane and axisymmetrical linearized problems of penetration of thin bodies into a plastic half-space were considered in [1] and [2].

In [3] linearized equations of some three-dimensional problems of the theory of ideal plasticity were considered.

In the present work, a problem of penetration of a thin blade into a plastic half-space is investigated. An analogy is established between the linearized theory of ideal plasticity and gas dynamics. This circumstance permits the utilization of the results obtained in the theory of a wing of finite span [4,5] for the determination of the velocity field and displacements for a problem of penetration of a thin blade into a plastic half-space.

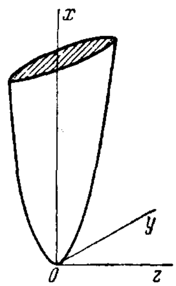
1. Consider a thin symmetrical blade being forced into a half-space of a rigid-plastic material (Fig. 1). The equation of the surface of a blade is given:

$$z = \delta F(x, y) \tag{1.1}$$

where δ is a small dimensionless parameter.

The half-space is determined initially by the inequality $x \geqslant 0$. We assume that the blade is stationary, and that the half-space moves vertically along the x-axis with constant velocity. The motion will be assumed sufficiently slow so that we may neglect small inertia effects. Denote by σ_{ij} the stress components; by ϵ_{ij} the components of the strain rate; by u, v, w the velocity components along the x-, y- and z-axes, respectively.

We seek the solution in the form:



$$\sigma_{ij} = \sigma_{ij}^{\circ} + \delta\sigma_{ij}', \qquad \varepsilon_{ij} = \varepsilon_{ij}^{\circ} + \delta\varepsilon_{ij}', \qquad u = u^{\circ} + \delta u', \ldots \tag{1.2}$$

The components with zero superscript correspond to the state $\delta = 0$, i.e. when the blade has zero thickness.

Let

Fig. 1.

$$\sigma_y^{\circ} = \sigma_z^{\circ} = -2k, \qquad \sigma_x^{\circ} = \tau_{xy}^{\circ} = \tau_{yz}^{\circ} = \tau_{zx}^{\circ} = 0$$
$$u^{\circ} = \text{const}, \qquad v^{\circ} = w^{\circ} = 0 \tag{1.3}$$

where k is the yield strength of a material.

The problem now is reduced to the determination of the components of the disturbances. These disturbances will be denoted by primed letters.

Let us assume that the plastic state is fully developed. Thus, we will have [6]

$$\begin{aligned} \tau_{xy}^2 &= \left(\sigma_x - \sigma + \tfrac{2}{3}k\right)\left(\sigma_y - \sigma + \tfrac{2}{3}k\right) \\ \tau_{yz}^2 &= \left(\sigma_y - \sigma + \tfrac{2}{3}k\right)\left(\sigma_z - \sigma + \tfrac{2}{3}k\right) \\ \tau_{zx}^2 &= \left(\sigma_z - \sigma + \tfrac{2}{3}k\right)\left(\sigma_x - \sigma + \tfrac{2}{3}k\right) \end{aligned} \tag{1.4}$$

or

$$\begin{aligned} \tau_{xy}\tau_{yz} &= \tau_{zx}\left(\sigma_y - \sigma + \tfrac{2}{3}k\right) \\ \tau_{zx}\tau_{xy} &= \tau_{yz}\left(\sigma_x - \sigma + \tfrac{2}{3}k\right) \\ \tau_{yz}\tau_{zx} &= \tau_{xy}\left(\sigma_z - \sigma + \tfrac{2}{3}k\right) \end{aligned} \tag{1.5}$$

Equalities (1.4) or (1.5) are obtained by elimination of the directional angles, using general relationships which are generalizations of the known relationships of M. Levy for the plane state of strain.

We note that (1.4) and (1.5) are equivalent. However, for the linearization both are used, since these relationships are squared.

Substituting (1.2) in (1.4) and (1.5), and taking into account (1.3), we obtain

$$\sigma_x' = \sigma_y' = \sigma_z' = \sigma', \qquad \tau_{yz}' = 0 \tag{1.6}$$

Setting

$$\sigma' = \frac{\partial U}{\partial x}, \qquad \tau_{xy}' = -\frac{\partial U}{\partial y}, \qquad \tau_{xz}' = -\frac{\partial U}{\partial z} \tag{1.7}$$

we obtain from the equilibrium conditions

$$-\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \qquad (1.8)$$

Consider now the relationship for the strain rates. We shall write down the basic relationship in the form [7]

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0 \qquad (1.9)$$

$$\varepsilon_x + \varepsilon_{xy}\frac{\sigma_y - \sigma + {}^2/_3 k}{\tau_{xy}} + \varepsilon_{xz}\frac{\sigma_z - \sigma + {}^2/_3 k}{\tau_{xz}} = \varepsilon_{xy}\frac{\sigma_x - \sigma + {}^2/_3 k}{\tau_{xy}} + \varepsilon_y +$$

$$+ \varepsilon_{yz}\frac{\sigma_z - \sigma + {}^2/_3 k}{\tau_{yz}} = \varepsilon_{xz}\frac{\sigma_x - \sigma + {}^2/_3 k}{\tau_{xz}} + \varepsilon_{yz}\frac{\sigma_y - \sigma + {}^2/_3 k}{\tau_{yz}} + \varepsilon_z$$

Linearizing (1.9) we obtain

$$\varepsilon_x' + \varepsilon_y' + \varepsilon_z' = 0, \qquad \varepsilon_{xy}' = \varepsilon_{xz}' = 0 \qquad (1.10)$$

It follows that

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} = 0, \qquad \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} = 0 \qquad (1.11)$$

Setting

$$u' = -\frac{\partial W}{\partial x}, \qquad v' = \frac{\partial W}{\partial y}, \qquad w' = \frac{\partial W}{\partial z}$$

we obtain

$$-\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \qquad (1.12)$$

In the following we shall consider the kinetics of the phenomena occurring during the penetration of a blade. The normal to the surface of a blade is

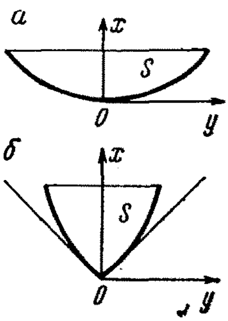


Fig. 2.

$$\mathbf{n} = \delta\frac{\partial F}{\partial x}\mathbf{i} + \delta\frac{\partial F}{\partial y}\mathbf{j} - \mathbf{k} \qquad (1.13)$$

where **i**, **j**, **k** are the coordinate unit vectors. The velocity vector is

$$\mathbf{v} = (u_0 + \delta u')\mathbf{i} + \delta v'\mathbf{j} + \delta w'\mathbf{k} \qquad (1.14)$$

On the surface of the blade the velocities are tangential to the blade, therefore

$$(\mathbf{v}\mathbf{n}) = \delta(u_0 + \delta u')\frac{\partial F}{\partial x} + \delta^2 v'\frac{\partial F}{\partial y} - \delta w' = 0 \qquad (1.15)$$

Linearizing (1.15) we obtain

$$w' = u_0 \frac{\partial F}{\partial x} \quad \text{for } z = 0, \text{ on } S$$

or

$$\frac{\partial W}{\partial z} = u_0 \frac{\partial F}{\partial x} \quad \text{for } z = 0, \text{ on } S \tag{1.16}$$

where S is the projection of the blade area on the xy-plane.

Figure 2 represents the penetration of a blade in the xy-plane ($z = 0$). Two cases may occur: (a) when the disturbances in the xy-plane do not extend beyond the region S; (b) when these disturbances extend beyond S.

The first case is realized for

$$-F_y / F_x \leqslant 1 \tag{1.17}$$

The second case is realized for

$$-F_y / F_x > 1 \tag{1.18}$$

In the second case we have to consider the conditions in the disturbed zone in the xy-plane outside of S. From the symmetry in this zone it follows that

$$W(x, y, -z) = -W(x, y, z)$$

Thus, in the disturbed zone in the xy-plane we have

$$W = 0 \quad \text{for } z = 0, \text{ outside of } S \tag{1.19}$$

For the determination of the bulging surface it is sufficient to know the velocity profile u'. Indeed, let us assume that the penetration depth is h. The equation of the bulging surface is obtained as

$$x - h = \delta\Phi(y, z) \tag{1.20}$$

The points on the surface of a half-space will be displaced by an amount $\delta s_x'$, $\delta s_y'$, $\delta s_z'$. The point with the coordinates $h + \delta s_x'$, $\delta s_y'$, $\delta s_z'$ must be, therefore, on the surface of the bulging material. We have

$$s_x' = \Phi(y + \delta s_y', \ z + \delta s_z') \tag{1.21}$$

Linearizing (1.21) we obtain

$$s_x' = \Phi(y, z) \tag{1.22}$$

The quantity $\delta s_x'$ represents the displacement of points of the surface along the x-axis.

In the theory of ideally plastic bodies the velocities derived from the basic relations are determined within a multiplicative constant. The true velocities of the disturbance along the x-axis are $\delta \lambda_u'$. Consequently

$$s_x' = \lambda \int_0^h u' dx \tag{1.23}$$

The quantity λ is found by equating the bulged volume of the material to the volume of that portion of the blade which penetrated into the body:

$$\lambda \int_0^h \int_\Sigma u' \, dx \, dy \, dz = \int_S F(x, y) \, dx \, dy \tag{1.24}$$

where Σ is the region of the determination of u' for $x = h$.

From (1.22) and (1.24) we obtain

$$\Phi(y, z) = \left(\int_0^h u' dx\right) \int_S F(x, y) \, dx \, dy \Bigg/ \left(\int_0^h \int_\Sigma u' dx \, dy \, dz\right) \tag{1.25}$$

2. Consider the first case (Fig. 2a). We write a solution in the form of a potential

$$W(x, y, z) = -\frac{u_0}{\pi} \int_q \frac{\partial F}{\partial x} \frac{d\xi \, d\eta}{\sqrt{(x-\xi)^2 - (y-\eta)^2 - z^2}} \tag{2.1}$$

where q is a portion of the xy-plane formed by its intersection with the characteristic cone with vertex at a point x, y, z.

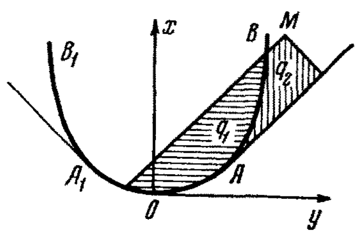


Fig. 3.

Note that the velocity potential is completely determined by the boundary conditions, which are independent of time. Thus, the velocity field is in a steady state. It is also clear that on the boundary between plastic and rigid regions we have $u' = v' = w' = 0$. The expression u' is found from (2.1):

$$u' = \frac{u_0}{\pi} \frac{\partial}{\partial x} \int_q \frac{\partial F}{\partial x} \frac{d\xi \, d\eta}{\sqrt{(x-\xi)^2 - (y-\eta)^2 - z^2}} \tag{2.2}$$

Consider now the second case (Fig. 2b). We utilize again (2.1). Outside of the blade, however, i.e. in the region q_2 (Fig. 3), the function $\partial F/\partial x$ must be zero. One circumstance has to be underlined. The analogy between the linearized relationships in gas dynamics and the theory of ideally plastic bodies is essentially complete, in spite of the fact that the basic relations in these two cases are completely different. In the first case reference is made to the irrotational flow of an ideal compressive gas and in the second to the flow (without shear along two components) of an incompressible ideally plastic material.

BIBLIOGRAPHY

1. Ivlev, D.D., O vdavlenii tonkogo lezviia v plasticheskuiu sredu (On the penetration of a thin blade into a plastic medium). *Izv. Akad. Nauk SSSR, OTN* No. 10, 1957.

2. Ivlev, D.D., O vdavlenii tonkogo tela vrashcheniia v plasticheskoe poluprostranstvo (On the penetration of a thin body of revolution into a half-space). *PMTF* No. 4, 1960.

3. Ivlev, D.D., Ob uravneniiakh linearizirovannykh prostranstvennykh zadach teorii ideal'noi plastichnosti (On the equations of the linearized three-dimensional problems of the theory of ideal plasticity). *Dokl. Akad. Nauk SSSR,* Vol. 130, No. 6, 1960.

4. Krasil'shchikova, E.A., *Krylo konechnogo razmakha v szhimaemom potoke (A Wing of Finite Span in a Compressible Flow).* Gostekhizdat, 1952.

5. Puckett, Supersonic wave drag of thin aerofoils. *J. Aeron. Sci.* Vol. 13, No. 9, 1946.

6. Ivlev, D.D., Ob obshchikh uravneniakh teorii ideal'noi plastichnosti v statike sypuchei sredy (On the general equations in the theory of ideal plasticity of the statics of a granular medium). *PMM* Vol. 22, No. 1, 1958.

7. Ivlev, D.D., O sootnosheniakh, opredeliaiushchikh plasticheskoe techenie pri uslovii plastichnosti Treska i ego obobshcheniakh (On the relationships determining plastic flow for Tresca plasticity condition and its generalization). *Dokl. Akad. Nauk SSSR* Vol. 124, No. 3, 1959.

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